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1976 J. Phys. A: Math. Gen. 9 145

(http://iopscience.iop.org/0305-4470/9/1/019)

J. Phys. A: Math. Gen., Vol. 9, No. 1, 1976. Printed in Great Britain. © 1976

A note on the Rarita–Schwinger equation in a gravitational background

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Received 7 July 1975

Abstract. It is shown how attempts to minimally couple the Rarita–Schwinger equation to a gravitational background break down.

1. Introduction

Recent work on particle creation in strong gravitational fields (eg Gibbons 1975) has been confined to scalar and spinor fields. It is easy to generalize to spin 1 with or without mass (ie the Maxwell or Proca fields). Beyond that there are no satisfactory theories. In this note we show how the minimally coupled Rarita–Schwinger (1941) formulation for spin $\frac{3}{2}$ breaks down. The results are similar to recent ones by Madore (1975) but the methods adopted are those of Velo and Zwanziger (1969). In addition, an explicit proof is given of the independence of the results of the free parameter which enters the theory.

2. Calculations

In flat space the Rarita–Schwinger equations are

$$(\mathrm{i}\gamma^\sigma\partial_\sigma + m)\psi^\alpha = 0 \tag{1a}$$

$$\partial_\sigma\psi^\sigma = 0 \tag{1b}$$

$$\gamma_\sigma\psi^\sigma = 0 \tag{1c}$$

where ψ^α is a 4-spinor vector with spinor index suppressed (our signature is $(+,-,-,-)$). Naïve minimal coupling by the replacement $\partial_\sigma \to \nabla_\sigma - \mathrm{i}eA_\sigma = \mathscr{D}_\sigma$ leads to immediate problems. Using the Ricci identity for spinor vectors

$$2\mathscr{D}_{[\mu}\mathscr{D}_{\nu]}\psi^\alpha = R^\alpha{}_{\sigma\mu\nu}\psi^\sigma + \tfrac{1}{4}\gamma^\epsilon\gamma^\sigma R_{\epsilon\sigma\mu\nu}\psi^\alpha - \mathrm{i}eF_{\mu\nu}\psi^\alpha \tag{2}$$

and the useful identity

$$R_{\alpha\beta\gamma\delta}\gamma^\beta\gamma^\gamma\gamma^\delta = -R^\sigma{}_{\alpha\sigma\beta}\gamma^\beta = -R_{\alpha\beta}\gamma^\beta \tag{3}$$

one obtains the equations

$$(\mathrm{i}\gamma^\sigma \mathscr{D}_\sigma + m)\mathscr{D}_\alpha \psi^\alpha = -\mathrm{i}(\tfrac{1}{2}R_{\alpha\beta} + \mathrm{i}eF_{\alpha\beta})\gamma^\alpha \psi^\beta \tag{5}$$

$$(\mathrm{i}\gamma^\sigma \mathscr{D}_\sigma + m)\gamma_\alpha \psi^\alpha = 0. \tag{6}$$

If one wishes to regard the set

$$(\mathrm{i}\gamma^\sigma \mathscr{D}_\sigma + m)\psi^\alpha = 0 \tag{7a}$$

$$\mathscr{D}_\alpha \psi^\alpha = 0 \tag{7b}$$

$$\gamma_\alpha \psi^\alpha = 0 \tag{7c}$$

as propagation equation and constraints respectively, one finds that the constraints will not in general be propagated off a Cauchy surface. Alternatively, if one imposes (7*a*), (7*b*) and (7*c*) one obtains a further algebraic constraint

$$(R_{\alpha\beta} + 2\mathrm{i}eF_{\alpha\beta})\gamma^\alpha \psi^\beta = 0.$$

Similar problems have been met by Buchdahl (1962) and Bell and Szekeres (1972) using a 2-component spinor approach. One solution has been proposed by Dowker (1967) which seems to involve non-local equations. An alternative approach is to follow Fierz and Pauli (1939) and seek a Lagrangian from which all three equations (1*a*, *b*, *c*) follow in flat space and minimally couple that. This usually involves auxiliary quantities (cf Hagen and Singh 1974), but for spin $\frac{3}{2}$ a simple 1-parameter family of Lagrangians can be found (Johnson and Sudarshan 1961). This is

$$L = \mathrm{i}\bar{\psi}^\mu \gamma^\sigma \partial_\sigma \psi_\mu + m\bar{\psi}^\mu \psi_\mu + \mathrm{i}(\alpha \bar{\psi}^\sigma \gamma_\sigma \partial_\alpha \psi^\alpha + \bar{\alpha}\bar{\psi}_\sigma \partial^\sigma \gamma_\alpha \psi^\alpha)$$
$$+ \mathrm{i}(\tfrac{3}{2}|\alpha + \tfrac{1}{3}|^2 + \tfrac{1}{3})\bar{\psi}^\sigma \gamma_\sigma \gamma^\alpha \partial_\alpha \gamma^\epsilon \psi_\epsilon - (\tfrac{1}{4} + |\alpha + \tfrac{1}{2}|^2)\bar{\psi}^\epsilon \gamma_\epsilon \gamma^\alpha \psi_\alpha, \tag{8}$$

provided $\alpha \neq -\frac{1}{2}$. If we now minimally couple (8) and consider the resulting Euler-Lagrange equations (with respect to variation of $\bar{\psi}^\alpha$ the Dirac adjoint) and contract with γ^α and $\mathscr{D}^\alpha$, one may obtain the constraints

$$\mathscr{D}_\sigma \psi^\sigma + \frac{s}{m} - \frac{3\bar{\alpha}+1}{2\bar{\alpha}+1}\frac{\mathrm{i}}{3}\gamma_\sigma \mathscr{D}^\sigma s = g = 0 \tag{9a}$$

$$\gamma_\sigma \psi^\sigma + \frac{2\mathrm{i}}{3}\frac{1}{2\bar{\alpha}+1}\frac{s}{m^2} = f = 0 \tag{9b}$$

with

$$s = \mathrm{i}\bar{\alpha}(\tfrac{1}{4}R + \mathrm{i}e\tfrac{1}{2}F_{\alpha\beta}\gamma^\alpha\gamma^\beta)\gamma_\sigma \psi^\sigma + \mathrm{i}(\tfrac{1}{2}R_{\alpha\beta} + \mathrm{i}eF_{\alpha\beta})\gamma^\alpha \psi^\beta. \tag{10}$$

Putting these constraints back into the Euler–Lagrange equation we obtain

$$(\mathrm{i}\gamma^\sigma \mathscr{D}_\sigma + m)\psi^\mu + \frac{2}{3}\frac{\mathscr{D}_\mu s}{m^2} - \frac{\bar{\alpha}+1}{3(2\bar{\alpha}+1)}\gamma^\sigma \mathscr{D}_\sigma \frac{\gamma_\mu s}{m^2} + \frac{\mathrm{i}}{3}\frac{3\bar{\alpha}+2}{2\bar{\alpha}+1}\frac{\gamma_\mu s}{m} = 0. \tag{11}$$

Note that each Lagrangian can be obtained from the $\alpha = 0$ case by the substitution

$$\psi^\alpha \to \psi^\alpha + \tfrac{1}{2}\alpha\gamma^\alpha \gamma_\sigma \psi^\sigma \tag{12}$$

as can the equations. From (11) one may deduce the equations

$$2\mathrm{i}g + mf - \mathrm{i}\gamma_\sigma \mathscr{D}^\sigma f = 0 \tag{13a}$$

$$\mathrm{i}\gamma_\sigma \mathscr{D}^\sigma g + mg - \mathrm{i}\bar{\alpha}f(\tfrac{1}{4}R + \mathrm{i}e\tfrac{1}{2}F_{\alpha\beta}\gamma^\alpha\gamma^\beta) = 0. \tag{13b}$$

Thus we see that (7*a*) will propagate the constraints off a Cauchy surface as required. However, if we seek the characteristic surfaces of (11), we find after some algebra that their normal n^α must obey

$$\det\left[(\gamma_\sigma n^\sigma)^3\gamma^\alpha n^\beta\left(g_{\alpha\beta}+\frac{1}{3}\frac{R_{\alpha\beta}}{m^2}-\frac{1}{6}Rg_{\alpha\beta}+\frac{2ie}{3m^2}\gamma^5F^*_{\alpha\beta}\right)\right]=0$$

where $4!\gamma^5=\epsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$ and $F^*_{\alpha\beta}=\frac{1}{2}\epsilon_{\alpha\beta\rho\sigma}F^{\rho\sigma}$. This is equivalent to the result of Madore (1975). In addition to the usual null surfaces there are 'anomalous' characteristics. If $eF_{\alpha\beta}=0$, these are the cones of the supplementary metric

$$a_{\alpha\beta}=g_{\alpha\beta}+(R_{\alpha\beta}-\tfrac{1}{2}Rg_{\alpha\beta})/(3m^2). \tag{14}$$

The fact that the characteristics are not null, leads to difficulties in the second quantized theory of the type first found by Johnson and Sudarshan (1961) and discussed in more detail by Velo and Zwanziger (1969). If Einstein's equations hold then (14) becomes

$$g_{\alpha\beta}+8\pi T_{\alpha\beta}/(3m^2)$$

which shows that the cones of the supplementary metric always lie outside the light cone, provided the stress tensor $T_{\alpha\beta}$ obeys the positive energy condition. The order of magnitude of the causality violation is

$$\frac{\delta v}{c}\simeq\frac{\text{density of matter}}{\text{nuclear density}},$$

which will be small except inside a neutron star or in the dense phase of the big bang. Velo and Zwanziger (1969) were able to show that the invariant sesquilinear form

$$\int J_\mu\,\mathrm{d}\Sigma^\mu=-\int(\bar{\psi}^\sigma\gamma_\mu\psi_\sigma+\alpha\bar{\psi}^\alpha\gamma_\alpha\psi^\mu+\bar{\alpha}\bar{\psi}^\mu\gamma_\alpha\psi^\alpha+\beta\bar{\psi}^\alpha\gamma_\alpha\gamma_\mu\gamma_\beta\psi^\beta)\,\mathrm{d}\Sigma^\mu$$

was positive definite, for data consistent with the constraints, on surfaces which lay outside both the light cone and the anomalous electromagnetic cones in flat space—it seems likely, but I have been unable to prove it, that this holds in the gravitational case. It is this fact which is responsible for difficulties in the second quantized theory. It is perhaps worth noting here that if one goes through the corresponding calculations for the Proca Lagrangian one finds that the characteristics remain null.

3. Conclusion

The fact that the simplest possible generalization of the flat space equations to curved space breaks down does not, of course, prove that a suitable generalization cannot be found. The experience of workers tackling the electromagnetic case where no suitable theory at present exists does not augur well. The opinions of workers in the field seem to be that a composite particle approach is more suitable to known particles of higher spin. Any attempt to prove that *no* simple field theory is not possible would need to tackle the problem of defining precisely what are the ingredients of such a theory.

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